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LETTER

Asymptotic throughput analysis of multicast transmission schemes

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Abstract

This paper demonstrates the use of extreme value theory in the asymptotic throughput analysis of wireless multicast and unicast for large number of users. Exact analysis of these schemes involves finding the probability distributions of maxima or minima of signal to noise ratios (SNRs). Jointly considering random user distribution and Rayleigh fading, exact distributions of these extreme values become complex and lack insights. On the other hand, asymptotic expressions obtained by using extreme value theory are quite accurate even for moderate number of users and they give insights about the performances without using simulations. The results of this analysis can be used in designing cellular unicast and multicast systems.

Keywords: Unicast; Multicast; Fading channel; Extreme value theory

1. Introduction

We consider a system where a base station (BS) has to transmit the same data to a set of nodes randomly located in a cellular area. In this case the BS can exploit the wireless multicast advantage by using an omnidirectional antenna so that all nodes in the communication range can hear the transmission. On the other hand the throughput performance of such a scheme is limited by the *worst* user in the group at any time. Depending on the cell radius it may be better to use unicast schemes where *best* user is chosen at each time (possibly based on channel conditions) so that multiuser diversity is exploited [1]. Analysis of both multicast and unicast transmission involves finding statistics of extreme (best or worst) channel conditions. In most cases such analysis results in complex expressions that are too complex to get insights [2,3].

Extreme value theory is useful in approximating the probability distributions of maxima or minima of a large number of random variables. It is therefore a practical tool in analyzing some channel-aware scheduling and transmission schemes [5]. In this paper we will use this tool in analyzing and comparing multicast and unicast transmission schemes. System model is as follows.

We consider a circular cell of radius d_c. There are N users independently and uniformly distributed in this cellular area. Therefore the probability density function of distance D_i of user i to the BS is as follows:

$$f_{D_i}(d) = \frac{2d}{d_\mathrm{c}^2}, \quad 0 < d \leqslant d_\mathrm{c}, \ \forall i \tag{1}$$

Path loss is modeled as $K D_i^n$, where n is the path loss exponent and K is a constant. Transmitted signals are also subject to Rayleigh fading, where the fading gain for transmitted power is exponentially distributed as

$$f_{H_i}(h) = h\,\mathrm{e}^{-h}, \quad h \geqslant, \forall i \tag{2}$$

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We assume that the r.v.'s D_i and H_i are i.i.d for all users $i = 1, \ldots, N$. Let S_i be the received signal to noise ratio (SNR) for user i, which is expressed as $S_i = P_{\rm BS}/N_0BK\,H_i/D_i^n$, $\forall i$, where $P_{\rm BS}$ is the BS power and N_0B is the noise power. We use the function $R_i(S_i) = B\log_2(1+\beta S_i)$ for the rate as a function of received SNR [6]. Here β can be related to the target bit error rate (BER) through the function $\beta = -1.5/\ln(5\,BER)$. Although this model is initially proposed for M-QAM systems, it is also successful in modeling the performance of continuous rate adaptation [7]. Now we define the scaled SNR $Z_i = \beta S_i$ as follows:

$$Z_i = \frac{P_{\rm BS}}{N_0BK}\frac{H_i}{D_i^n}\beta = \frac{\rho_0 H_i}{L_i}, \quad \forall i \tag{3}$$

Here we define $\rho_0 = P_{\rm BS}/N_0BK\beta$ and $L_i = D_i^n$. The cumulative distribution function (c.d.f.) of L_i is $f_{L_i}(l) = (2/nr_{\rm c}^2)l^{2/n-1}$, $\forall i$. C.d.f. for received SNR in Rayleigh fading and random user distribution was derived in [8] as follows:

$$F_{Z_i}(z) = 1 - \frac{2}{n}\left(\frac{zd_{\rm c}^n}{\rho_0}\right)^{-2/n}\gamma\left(\frac{2}{n}, \frac{zd_{\rm c}^n}{\rho_0}\right), \quad \forall i \tag{4}$$

Incomplete gamma function, $\gamma(.,.)$, is as follows:

$$\gamma\left(\frac{2}{n}, \frac{zd_{\rm c}^n}{\rho_0}\right) = \int_0^{zd_{\rm c}^n/\rho_0} u^{2/n-1}\,{\rm e}^{-u}\,{\rm d}u \tag{5}$$

In this paper we analyze a system , in which the same data have to be transmitted to all nodes in the system. We consider two alternative ways to perform this operation, multicasting and unicasting. Below, we present the asymptotic performance analysis of both schemes.

2. Asymptotic multicast analysis

In order to find the probability distribution function for the multicast rate, we need to find the distribution of the minimum of the rates in all users. Minimum rate corresponds to the minimum scaled SNR,

$$Z_{\min,N} = \min_{i=1,\ldots,N}(Z_i) \tag{6}$$

Using standard techniques to find the c.d.f. of the minima results in lengthy expressions, which also lack insights [3]. It is reasonable to use extreme value theory to find asymptotic approximations to the c.d.f. of minima. We use the following theorem:

Theorem 1 (Castillo [4]). *Let Z_i, $i = 1, \ldots, N$ be i.i.d. random variables with c.d.f. $F_Z(z)$. Let $\alpha(F)$ be defined as $\alpha(F_Z) = \inf\{z : F_Z(z) > 0\}$. If $\alpha(F) > -\infty$ and the function*

$$F_Z^*(z) = F_Z\left(\alpha(F) - \frac{1}{z}\right); \quad z < 0 \tag{7}$$

satisfies

$$\lim_{t\to-\infty}\frac{F_Z^*(tz)}{F_Z*(t)} = z^{-\gamma}, \quad \gamma > 0 \tag{8}$$

then there exist constants c_N and d_N such that $(Z_{\min,N} - c_N)/d_N$ converges to (in other words, $F_Z(z)$ lies in the domain of attraction *of) Weibull distribution for minima, where the normalizing constants are $c_N = \alpha(F)$ and $d_N = F_Z^{-1}\left(\frac{1}{N}\right) - \alpha(F)$.*

Weibull distribution (minima) is $L_{2,\gamma} = 1 - \exp(-x^\gamma)$ if $x > 0$ and 0 otherwise [4]. *It is equivalent to exponential distribution for $\gamma = 1$.*

By using Taylor expansion we can write the c.d.f. in (4) as

$$F_Z(z) = 1 - \frac{2}{n}\sum_{i=0}^{\infty}\left(\frac{zd_{\rm c}^n}{\rho_0}\right)^i\frac{(-1)^i}{i!(i+2/n)} \tag{9}$$

If we apply (7) and (8) to the c.d.f. in (9) we obtain,

$$\lim_{t\to-\infty}\frac{F_Z^*(tz)}{F_Z*(t)} = \lim_{t\to-\infty}\frac{1 - 2/n\sum_{i=0}^{\infty}\left(\frac{d_{\rm c}^n}{tz\rho_0}\right)^i\frac{(-1)^i}{i!(i+2/n)}}{1 - 2/n\sum_{i=0}^{\infty}\left(\frac{d_{\rm c}^n}{t\rho_0}\right)^i\frac{(-1)^i}{i!(i+2/n)}} \tag{10}$$

$$\lim_{t\to-\infty}\frac{F_Z^*(tz)}{F_Z*(t)} = z^{-1} \tag{11}$$

where (10) is obtained by using the Taylor expansion (${\rm e}^u = 1 + u + u^2/2! + \cdots$) and (11) is obtained by applying L'Hospital's rule once. Therefore $\gamma = 1$ and the minima converges in distribution to an exponential random variable. Normalizing constant c_N is zero since $\alpha(F) = 0$. The other constant is $d_N = F_Z^{-1}(1/N) = Z_{1/N}$. Finding the exact value is hard because of the structure of the function. However, since we are interested in large N (small $1/N$), corresponding $Z_{1/N}$ is small, we can apply Taylor approximation. We can find an approximate d_N as follows:

$$\frac{1}{N} = 1 - \frac{2}{n}\sum_{i=0}^{\infty}\left(\frac{Z_{1/N}d_{\rm c}^n}{\rho_0}\right)^i\frac{(-1)^i}{i!(i+2/n)} \tag{12}$$

$$\frac{1}{N} \simeq 1 - 1 + \frac{2}{n}\frac{Z_{1/N}d_{\rm c}^n}{\rho_0}\frac{1}{(1+2/n)} \tag{13}$$

$$d_N = Z_{1/N} \simeq \frac{\rho_0}{d_{\rm c}^n}\left(\frac{n+2}{2N}\right) \tag{14}$$

Therefore the limiting distribution for the minimal scaled SNR is $\lim_{N\to\infty}F_{Z_{\min,N}}(z) \simeq 1 - {\rm e}^{-z/d_N}$, where d_N is as in (14).

2.1. Multicast rate distribution

Note that $R_i(Z_i) = B\log_2(1+Z_i)$ also takes values from zero to infinity and it is a monotonic increasing function of

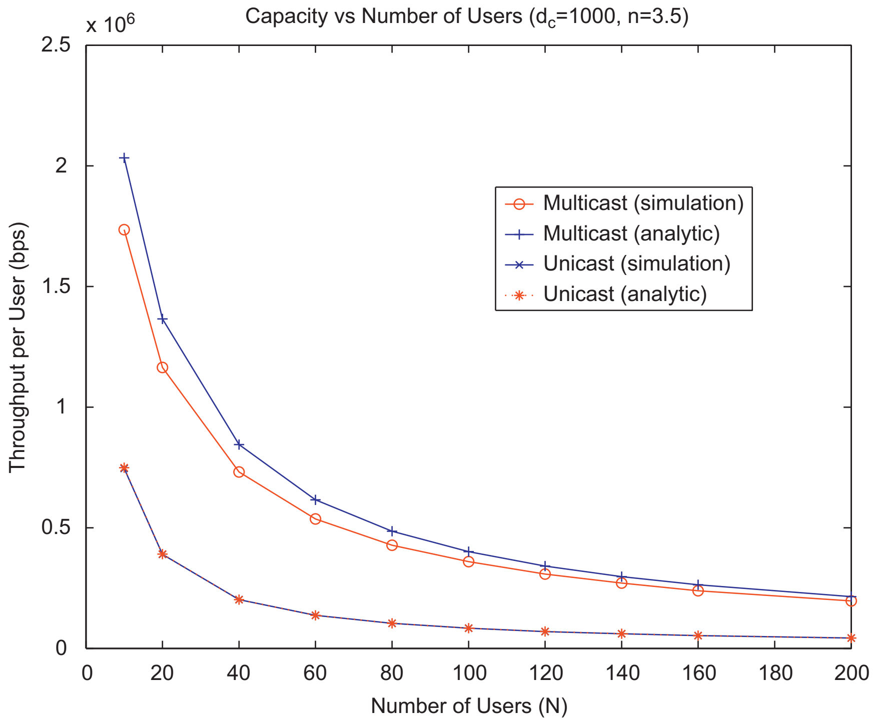

Fig. 1. Per user throughput for multicast and unicast schemes versus number of users. Path loss exponent is $n = 3.5$.

Z_i. Its c.d.f. is $F_R(r) = F_Z(2^{r/B} - 1)$. If we apply (7) and (8) and substitute $1/tz$ by $2^{1/trB} - 1$ and $1/t$ by $2^{1/tB} - 1$ in (10) using L'Hospital's rule once we find that the r.v. for multicast rate $R_{\min,N} = B\log_2(1 + Z_{\min,N})$ also belongs to the domain of attraction of Weibull (more specifically exponential since $\gamma = 1$ again) distribution. The normalizing constants are $c_N = 0$ and $d_N = B\log_2(1 + F_Z^{-1}(1/N))$. Per user multicast capacity is therefore approximately,

$$R_{\mathrm{M}} \simeq B\log_2\left(1 + \frac{\rho_0}{d_{\mathrm{c}}^n}\left(\frac{n+2}{2N}\right)\right) \tag{15}$$

Since this rate reaches to all users simultaneously, total multicast throughput is $N \times R_{\mathrm{M}}$. For large N (small SNR), from the approximation $N \times B\log_2(1 + \rho_0/d_{\mathrm{c}}^n((n+2)/2N)) \simeq N \times \rho_0/d_{\mathrm{c}}^n((n+2)/2N)$ we see total multicast throughput saturates as the number of users increases.

3. Asymptotic unicast analysis

We consider a unicast scheme, where at each time interval the BS chooses the user that maximizes normalized received SNR, $i^* = \operatorname{argmax}_{i=1,\ldots,N} \frac{S_i}{\overline{S_i}}$, where $\overline{S_i}$ is the time averaged SNR. Such a scheme both exploits multiuser diversity and maintains proportional fairness among nodes by normalization. We consider a system where users experience Rayleigh fading that is i.i.d. for each user and each time slot. Therefore average SNR $\overline{S_i}$ reflects the long-term channel gain, which is the path loss in our case. Thus, the normalized SNR reflects the current multipath fading gain. Maximum normalized SNR $\frac{S_{i*}}{\overline{S_{i*}}}$ therefore means the maximum $H_{\max,N} = H_i^* = \max_{i=1,\ldots,N} H_i$ and it is independent of the distances. Let D_{i*} be the distance of the chosen user i^*. Then the rate of the chosen user is $R_{i*} = B\log_2\left(1 + \frac{\rho_0 H_{\max,N}}{D_{i*}^n}\right)$. In [5] it was found using extreme value theory that for large N, R_{i*} can be closely approximated as $R_{i*} \simeq B\log_2\left(1 + \frac{\rho_0 \ln N}{D_{i*}^n}\right)$. We can find the approximate total system throughput by taking the expectation of this rate with respect to the distance from the BS.

$$\begin{aligned} R_{\mathrm{U}} &\simeq \int_{d=0}^{d=d_{\mathrm{c}}} B\log_2\left(1 + \frac{\rho_0 \ln N}{d^n}\right)\frac{2d}{d_{\mathrm{c}}^2}\, d_d \\ &= B\frac{d^2}{d_{\mathrm{c}}^2}\log_2\left(1 + \frac{\rho_0 \ln N}{d^n}\right)\Bigg|_0^{d_{\mathrm{c}}} \\ &\quad + \int_0^{d_{\mathrm{c}}} \frac{Bnd\rho_0 \ln N/\ln 2}{d_{\mathrm{c}}^2(d^n + \rho_0 \ln N)} \end{aligned} \tag{16}$$

For large N ($\rho_0 \ln N \gg d_{\mathrm{c}}^n$) we can approximate the term inside integral as $Bnd/d_{\mathrm{c}}^2 \ln 2$. In that case we can approximate the total unicast throughput as

$$R_{\mathrm{U}} \simeq B\log_2\left(1 + \frac{\rho_0 \ln N}{d_{\mathrm{c}}^n}\right) + \frac{Bn}{2\ln 2} \tag{17}$$

Since each user has equal chance of being selected $(1/N)$, per user unicast capacity is R_{U}/N.

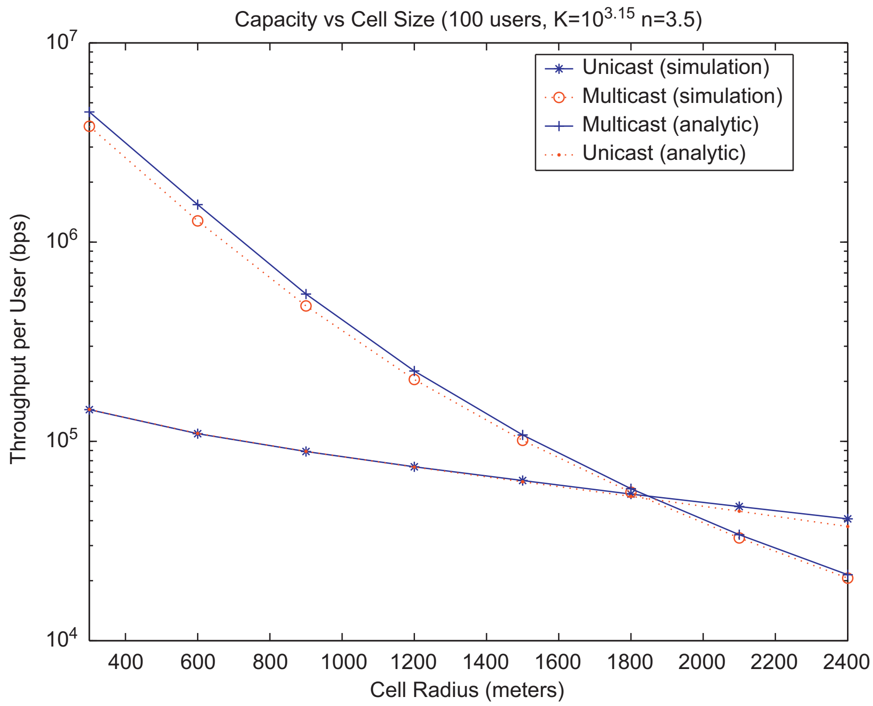

Fig. 2. Per user capacities for multicast and unicast schemes versus cell radius: $K = 10^{3.15}$, $n = 3.5$.

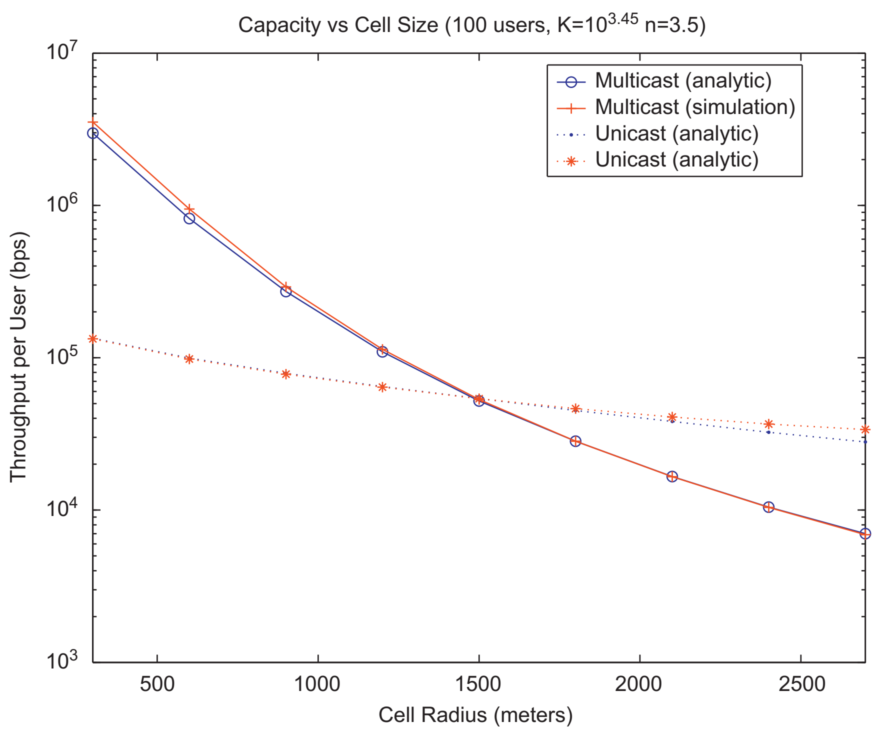

Fig. 3. Per user capacities for multicast and unicast schemes versus cell radius: $K = 10^{3.45}$, $n = 3.5$.

4. Performance evaluation

We evaluated both transmission schemes and accuracy of the asymptotic analysis by simulations. We consider typical broadband wireless access system parameters [9]. We use β=0.2, which corresponds to a target BER of 1.1×10^{-4}. We perform the simulations in MATLAB and take the average of 100 000 samples for each point in the simulation results.

Fig. 1 shows the unicast and multicast per user throughput for number of users varying from 10 to 200 for the path loss exponent $n=3.5$ and path loss constant $K=10^{3.15}$ (Suburban Macro NLOS channel model [9]). Cell radius is $d_c=1000$ m and channel bandwidth is $B=1$ MHz. We observe that analytic expressions become more accurate as number of users increase. Analytic results for unicast is almost identical to the simulation results. Multicasting is better than unicasting for this set of parameters.

Fig. 2 illustrates per user throughputs for both schemes versus cell sizes varying from 300 to 2400 m for the same path loss parameters. We observe as in [3] that unicast scheme becomes more advantageous in terms of throughput if the cell radius is greater than a threshold. We also observe that the simple throughput expressions that we found are very accurate.

Fig. 3 compares both schemes for varying cell radius. This time we consider an urban macrochannel model [9] with $K=10^{3.45}$. This is a more severe path loss and as we see in the figure unicast scheme becomes more advantageous even in smaller cell sizes. Again, analytic results closely follow the simulations.

5. Conclusion

In this work we demonstrated the use of extreme value theory in analyzing the throughput performance for multicast and unicast transmission schemes. We see by simulations that asymptotic expressions are quite accurate even for moderate number of users. Results can be used in choosing the appropriate transmission scheme (unicast or multicast) easily based on the cell size. Results can also be extended in order to find and analyze more advanced multicasting schemes such as grouping the users at cell edge separately so that they do not affect the throughput of near users. Solutions like layered multicast can also be tried so that near users receive better quality data. Asymptotic analysis of such more complex schemes is a subject for future work.

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