# Optimal Power Control for Wireless Queueing Networks

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*Abstract* — **In this paper, we address the problem of power control in wireless queueing networks. We consider a discrete time (slotted) wireless system of single link. Data arrive in packets according to a Bernoulli process of rate $\lambda$. A simple ARQ (send and wait) retransmission policy is used for the transmission of packets. The control parameter in this system is the transmission power of a packet, which is decided at every time slot if there is a packet to be transmitted at the queue. We consider choosing dynamically from two power levels dependent on the system state (number of packets in the system and previous decision). Our goal is to jointly minimize energy expenditure and queue overflow. Using the dynamic programming framework, we prove that the optimal policy that chooses one of the two power levels is of threshold type, that is, the optimal policy switches from small to large and large to small power at certain queue length states.**

## I. Introduction and System Model

A key concern in wireless networks is the energy efficiency. Limited and non-renewable battery supplies in most of the wireless devices led to some adaptive transmission schemes that efficiently use these resources. Power control is one of those adaptive schemes. The main motivation in the past work on power control was mitigating the effects of interference in order to maximize the achievable capacity (e.g. [1], [2]). Recently the work in [3] addressed the problem of minimizing energy expenditure subject to a deadline or delay constraint over a wireless link by varying transmission power. In [5], the authors aim at designing a transmission schedule that maximizes battery lifetime, subject to some delay constraints. They were able to do this by transmitting at reduced powers over longer durations, and exploiting the electro-chemical mechanisms in the batteries.

There are mainly two issues of primary interest in communications [4] : 1) Random message arrivals to the transmitters and limited buffer capacities, 2) The noise and interference that affect the success of transmissions. These two aspects of communication were often treated separately in the previous work. However they should be treated together in order to increase the communication performance. Having these concerns, this paper considers a wireless link with random message arrivals and a finite capacity buffer, and addresses the problem of jointly minimizing energy expenditure and queue overflow by dynamically choosing from two alternative power levels. We prove by using the Dynamic Programming framework that the optimal strategy of selecting transmission power is of threshold type. Previously the works in [6], [7] and [8] used the Dynamic Programming approach in proving the optimality of threshold policies for different problems. In [7] and [8] the optimal control of a queueing system with two heterogeneous servers is considered, whereas in [6], the authors address the problem of modulation level control in a fading environment.

The general assumptions on the system considered in this work is as follows. A source supplies a wireless node with packets that arrive according to a Bernoulli process of rate $0 < \lambda < 1$ packets/time slot. Initially we are assuming that the number of bits per packet (B) is equal to one. The arriving packets enter in the input queue that has a capacity of K packets, where the bits arriving in excess of K packets are lost. Packets (bits) are transmitted by using BPSK modulated signals of duration $T_s$, where $T_s$ is also equal to the time slot length ($R_s = 1/T_s$ is the symbol rate). $P$ is the power that is used in transmission, and $PT_s$ is the energy per single transmission.

A simple ARQ retransmission protocol is assumed to be used. There is an error-free feedback channel with zero feedback delay and unless all of the symbols of the transmitted packet are error-free, they are retransmitted. The number of retransmissions is therefore geometric distributed. Received signal is subject to an additive white gaussian noise of power spectral density $N_0$. Therefore, the probability of successful packet transmission is equal to $\mu = 1 - Q\left(\frac{PT_s}{N_0}\right)$ and the expected transmission duration is $1/\mu$ time slots. Expected energy expenditure during one packet transmission is therefore $PT_s 1/\mu$. We can make the following initial comments:

1. Smaller transmission power means smaller energy/per single transmission.
2. However, smaller power has also greater average service time, therefore during the service of a packet, more overflows occur.
3. Large power can be advantageous for greater queue sizes; energy per packet as well as overflow during packet service can be smaller.

In Section II, we will consider the problem of dynamically choosing between two alternative power levels, $P_1$ and $P_2$ ($P_1 < P_2$). The decision instants will be the new packet transmission epochs and the same power level will be used for all od the retransmissions of a packet. We will define single stage costs incorporating energy and overflow objectives and find the state transition probabilities at each time slot according to the Dynamic Programming Framework. Our ultimate goal is to see whether, or, for what conditions the optimal policy is of threshold type. The optimality of threshold policy and the optimal threshold can change with the alternative power levels, channel condition, switching cost and the relative weights of energy expenditure and overflow.

## II. Markov Decision Process

In order to do describe the system in the form of a Markov Chain, we define two variables, which are the system occupancy ($x_t \in \{0, 1, \ldots, K\}$) and the service occupancy ($u_t \in \{0, 1, 2\}$) at time slot $t$ after the decision for that time slot is made. Service occupancy $u = 0$ means that the transmitter was idle in the $t^{th}$ slot. $u = 1$ ($u = 2$) means that transmission power $P_1$ ($P_2$) is used. We also define feedback ($f_t$), where $f_t = 1$ means that transmission in the time slot $t$ is successful. $U(x, u, f) \subset \{0, 1, 2\}$ is the set of available control actions when the system state is $(x, u)$ and feedback $f$ is received. Table 1 shows the allowable control actions corresponding to each state and feedback value. If a positive feedback is received, control actions $u_{t+1} = 1$ and 2 are allowed for $(x_{t+1} \geq 1)$, and $u_{t+1} = 0$ is allowed only for $x_t = 0$. If a negative feedback is received, only $u_t = s_t$ is allowed (The same packet will be retransmitted with same power).

| $x_t$ and $f_t$ | Feasible Actions ($U(x_t, u_t, f_t)$) |
|---|---|
| $f_t = 0$ | $U(x_t, u_t, f_t) = \{u_t\}$ |
| $x_t \geq 1$, $f_t = 1$ | $U(x_t, u_t, f_t) = \{1, 2\}$ |
| $x_t = 0$ | $U(x_t, u_t, f_t) = \{0\}$ |

Tab. 1: Feasible (Allowable) control sets corresponding to all state values.

We define a set ($\Gamma$) of policies, such that when transmitting the head of the line packet for the first time at time $t + 1$, a policy $\gamma \in \Gamma$ uses the information $\{x_t, u_t, f_t\}$, in order to select the transmission power between $P_1$ or $P_2$. A policy $\gamma^*$ is optimal among the policies in $\Gamma$ if it minimizes the discounted cost function,

$$V_\alpha = E_x^\gamma \sum_{t=0}^{\infty} \alpha^t g(x_t, u_t) \quad (1)$$

for all $x_t \in X$ and $u_t \in U(x_{t-1}, u_{t-1}, f_{t-1})$ where $E_x^\gamma$ is the expectation with respect to $x$ when policy $\gamma$ is used. The cost in Equation (1) is minimized in order to jointly minimize the energy expenditure and queue overflows. The function $g(x_t, u_t)$ is the one stage cost when the number of bits in the system is $x_t$ and control $u_t \in U(x_{t-1}, u_{t-1}, f_{t-1})$ is applied. It has to combine the energy expenditure and bit overflow in a single stage. The term $\alpha$ is the discount factor and it is the probability that the communication session ends at the end of the time slot. Now we will first define the single stage cost functions.

### A. Single Stage Cost

Single stage cost is the weighted sum of energy expenditure and overflow in a single stage. Energy cost is proportional to the energy expenditure in a time slot. Overflow cost is proportional to the total bits that come in excess of the queue size during a time slot. We assume that number in the system is decremented by one, when the packet is successfully transmitted.

Overflow Cost: Now we will derive the expected number of overflowed bits in a stage (slot). Let $O(x, u)$ be the expected overflow in a time slot where number in system is $x$ and control $u$ is applied. Let $\lambda$ be the packet arrival probability during the time slot. The overflow cost is equal to:

$$O(x, u) = \begin{cases} 0, & x < K \\ \lambda, & x = K \end{cases} \quad (2)$$

Energy Expenditure Cost: If power level $P_i$ is used, then energy expenditure in a time slot is equal to $P_i T_{slot}$. Thus energy cost ($E(x, u)$) is given by,

$$E(x, u) = \begin{cases} P_u T_s, & u = 1, 2 \\ 0, & u = 0 \end{cases} \quad (3)$$

For simplicity, we omit the $T_s$ parameter from the energy cost, since it is a constant. The overall one stage cost is the weighted sum of energy expenditure and overflow, which is written as follows:

$$g(x, u) = c_1 E(x, u) + c_2 O(x, u), \forall x \in X, u \in U \quad (4)$$

where $c_1$ and $c_2$ are the weights of energy and overflow cost.

### B. State Transitions

We define $d_{i,(j,f)}(u)$ as the probability that number of bits in the system changes from $i$ to $j$, and feedback $f$ is received. We can write $d_{i,(j,f)}(u)$ as follows:

$$d_{i,(j,f)}(u) = \begin{cases} 1-\lambda & i = 0, j = 0, f = 1, \\ \lambda & i = 0, j = 1, f = 1, \\ \lambda\mu_u & K > i > 0, j = i, f = 1, \\ (1-\lambda)(1-\mu_u) & K > i > 0, j = i, f = 0, \\ \lambda(1-\mu_u) & K > i > 0, j = i+1, f = 0, \\ (1-\lambda)\mu_u & K > i > 0, j = i-1, f = 1, \\ 1-\mu_u & i = K, j = i, f = 0, \\ \mu_u & i = K, j = i-1, f = 1, \\ 0 & |j-i| > 1. \end{cases} \quad (5)$$

Here $\mu_u$ is the success probability corresponding to power $P_u$. We assume that feedback 1 is received if no packet is transmitted. The Dynamic Programming (DP) equation can be written as follows:

$$V^0(x, u) = g(x, u) \quad (6)$$

$$V^{n+1}(x, u) = g(x, u) + \alpha \sum_{x', f} d_{x,(x',f)}(u) W^n(x', u, f) \quad (7)$$

Here, in order to simplify the analysis, we use $W^n$ as an intermediate step as the cost-to-go just before new decision is made. $W^n(x, u, f)$ is defined as follows:

$$W^n(x, u, f) = \min_{u' \in U(x,u,f)} V^n(x, u') \quad (8)$$

where $U(x, u, f)$ is the feasible set of actions corresponding to state $(x, u)$ and feedback $f$, which is described in Table 1. $V^{n+1}(x, u)$ can be written more clearly as follows:

$$V^{n+1}(0, 0)|_{x=0} = g(x, 0) + \alpha[\lambda W^n(1, 0, 1) + (1-\lambda) W^n(0, 0, 1)] \quad (9)$$

$$V^{n+1}(x, u)|_{0<x<K} = g(x, u) + \alpha[\lambda\mu_u W^n(x, u, 1) + (1-\lambda)\mu_u W^n(x-1, u, 1) + \lambda(1-\mu_u) W^n(x+1, u, 0) + (1-\lambda)(1-\mu) W^n(x, u, 0)] \quad (10)$$

$$V^{n+1}(K, u)|_{x=K} = g(x, u) + \alpha[\mu_u W^n(K-1, u, 1) + (1-\mu_u) W^n(K, u, 0)] \quad (11)$$

## III. Optimality of a Threshold Policy

We will now show that the optimal policy is of threshold type. If $V^n(x, 1) - V^n(x, 2)$ is positive (negative) choosing $P_2$ ($P_1$) is optimal. Therefore it can be helpful to see the characteristic of $V^n(x, 1) - V^n(x, 2)$ changing with the number ($n$) of iterations. Figures 1 and 2 are generated for $c_1 = 1$, $c_2 = 10$, buffer capacity $K = 20$ and noise PSD $N_0 = 1 \times 10^{-4}$. Figure 1 shows $h_x^n = V^n(x, 1) - V^n(x, 2)$ for $x = 0, \ldots, K$ and for $n = 0, \ldots, 4$. It can be observed that $V^n(x, 1) - V^n(x, 2)$ is negative and decreasing first, then it increases and crosses zero, where $P_2$ becomes the optimal power. $V^n(x, 1) - V^n(x, 2)$ is not a monotone increasing function because the system is discrete-time and at $x=0$ no transmission is made and no energy is spent.

Figure 2 shows the optimal power for $x=0,\dots,K$ and the optimal cost to go $V(x,1)$ again for $0\le x\le K$. In the first graph of Figure 2 we observe the threshold characteristics. In the second graph we see that, $V^n(x,1)$ is first concave (when $V^n(x,1)-V^n(x,2)$ is decreasing), and then it becomes convex (when $V^n(x,1)-V^n(x,2)$ is increasing).

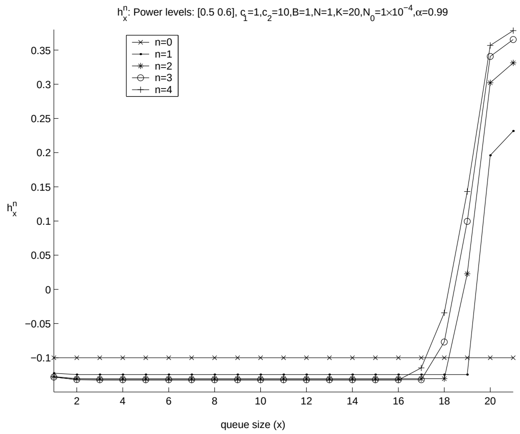


Fig. 1: $h_x^n=V^n(x,1)-V^n(x,2)$ for n= 0 to 4.

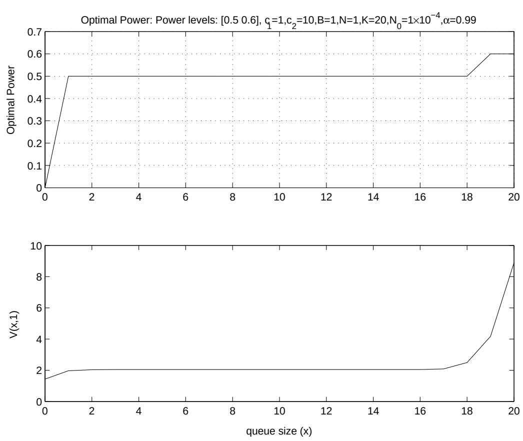


Fig. 2: The optimal power and optimal cost-to-go vs. number in system,$x$.

To prove the optimality of threshold policy, it is sufficient to show that for every iteration $n$:

1. For $x$ greater than or equal to some $x^n\in\{0,1,\dots,K\}$:

$$V^n(x,1)-V^n(x,2)\le V^n(x+1,1)-V^n(x+1,2) \quad (12)$$

2. For $x<x^n$,

$$0>V^n(x,1)-V^n(x,2)\ge V^n(x+1,1)-V^n(x+1,2) \quad (13)$$

This means that at every iteration $V^n(x,1)-V^n(x,2)$ is a negative and decreasing function for x smaller than some $x^n$. For $x\ge x^n$ it is a strictly increasing function of $x$. If eventually $V^n(x,1)-V^n(x,2)$ becomes greater than zero then using $P_2$ becomes optimal.

**Theorem 1** *Let's first make the following definitions:*
$h_x^n=V^n(x,1)-V^n(x,2)$,
$k_x^n=V^n(x+1,1)-V^n(x,1)$ *and*
$l_x^n=V^n(x+1,2)-V^n(x,2)$.
*If $P_1<P_2$, the below inequalities hold in every iteration $n$:*

1. *For $x$ greater than or equal to some $x^n\in\{0,1,\dots,K\}$:*

$$\begin{aligned} h_x^n &\le h_{x+1}^n, &(14)\\ k_x^n &\le k_{x+1}^n, &(15)\\ l_x^n &\le l_{x+1}^n, &(16)\\ \text{and}\ \ 0 &\le k_x^n, l_x^n, \forall x. &(17)\end{aligned}$$

2. *For $x<x^n$:*

$$\begin{aligned} 0>h_x^n &\ge h_{x+1}^n, &(18)\\ k_x^n &\ge k_{x+1}^n, &(19)\\ l_x^n &\ge l_{x+1}^n, &(20)\\ \text{and}\ \ 0 &\le k_x^n, l_x^n, \forall x. &(21)\end{aligned}$$

*Inequalities (14) and (18) are the same as (12) and (13),respectively. The inequalities of $k_x^n$ and $l_x^n$ are about the convexity and concavity $V^n(x,1)$ and $V^n(x,2)$ and they are necessary for the first inequality to hold. Finally last inequality means that $V^n(x,1)$ and $V^n(x,2)$ are increasing in $x$.*

**Proof** See the Appendix.

## IV. Practical Determination of the Threshold

The application of the optimal policy requires the computation of the threshold. Once we proved that the optimal policy of threshold type, we can now find more practical ways to determine the threshold value, instead of doing iterations. Let $0\le t\le K$ be the threshold value. We will find the queue state probabilities $\bar{\pi}=[\pi_0,\pi_1,\dots,\pi_K]$ at the departure (transmission completion) epochs, by using the imbedded Markov chain analysis [9]. For queue sizes smaller than $t$, $P_1$ will be used, whereas for $x\ge t$, $P_2$ will be used. Therefore transition probability ($\Pi_{ij}^t$) from state $i$ to $j$ for threshold $t$ is as follows:

$$\Pi_{ij}^t=\begin{cases} a_j^1, & i=0, j=0,1,\dots,K-1,\\ a_{j-(i-1)}^1, & i=1,2,\dots,t-1, j\ge i-1,\\ a_{j-(i-1)}^2, & i=t,t+1,\dots,K, j\ge i-1,\\ r_{K-1}^1, & i=0, j=K,\\ r_{K-i}^1, & i=1,2,\dots,t-1, j=K,\\ r_{K-i}^2, & i=t,2,\dots,K, j=K,\\ 0, & else.\end{cases} \quad (22)$$

where $a_j^u$ is the probability that j arrivals occur during a service period of a packet if transmission power is $P_u$. We can formulate $a_j^u$ as:

$$a_j^u=\sum_{k=max(1,j)}^{\infty}(1-\mu_u)^{k-1}\mu_u\binom{k}{j}\lambda^j(1-\lambda)^{k-j} \quad (23)$$

and $r_j^u=\sum_{i=j+1}^{\infty}a_i^u, j\ge 0$. After continuing the derivation from Equation (23), we find that $a_j$ can be written as follows:

$$a_j^u=\begin{cases} X^uZ^u, & j=0\\ X^u(Y^u)^j, & j>0\end{cases} \quad (24)$$

where $Z^u=(1-\lambda)(1-\mu_u)$, $X^u=\frac{\mu_u}{(1-\mu_u)(1-Z^u)}$ and $Y^u=\frac{\lambda(1-\mu_u)}{(1-Z^u)}$. We can find the queue state probabilities corresponding to threshold $t$ by solving the equation $\bar{\pi}^t\Pi^t=\bar{\pi}^t$.

Average cost per successfully transmitted packet is the weighted sum of the average energy per successful transmission and average overflowed packets during a successful transmission. First we will derive the expected number of overflowed bits per successful transmission. Let $O(x,u)$ be the expected overflow in a transmission period that starts with queue size $x$, where power $P_u$ is used. $O(x,u) = \sum_{j=K-x+1}^{\infty}\sum_{k=j}^{\infty}(1-\mu_u)^{k-1}\mu_u \begin{pmatrix} k \\ j \end{pmatrix} \lambda^j(1-\lambda)^{k-j}(j-K+x)$. After making some manipulations we find:

$$O(x,u) = \frac{\lambda}{\mu_u}\left(\frac{\lambda(1-\mu_u)}{1-(1-\lambda)(1-\mu_u)}\right)^{K-x} \qquad (25)$$

The expected energy cost during a successful transmission period is equal to $E(x,u) = \frac{P_u}{\mu_u}$, where $P_u$ and $\mu_u$ are the transmission power and success probability corresponding to control action $u$, respectively. The overall cost $R(x,u)$ is the weighted sum of energy expenditure and overflow:

$$R(x,u) = \begin{cases} c_1E(x,u) + c_2O(x,u) & ,x > 0 \\ c_1E(1,u) + c_2O(1,u) & ,x = 0 \end{cases} \qquad (26)$$

Then we can search for the optimal threshold $t \in \{0, 1, \ldots, K\}$ that minimizes the following average cost per packet transmission:

$$t^* = argmin_t\{\sum_{x=1}^{t-1}\pi_x^t R(x,1) + \sum_{x=t}^{K}\pi_x^t R(x,2)\} \qquad (27)$$

This method is based on a priori calculation of $R(x,u)$ values for all queue size $x$ and $u = 1$ and 2.

## V. Some Computational Results

In this section we will present some simulation results for the solution of the power control problem posed in (7). In these simulations, value iteration is used to solve (7). Simulation results indicate that the optimal strategy for selecting the rates is indeed a threshold policy. In all of our simulations, the buffer capacity is taken as $K = 20$, and noise power spectral density is equal to $N_0 = 1 \times 10^{-4}$.

We first consider the effect of changing the power values on the threshold value. Figure 3 shows the optimal power control policies corresponding to $(P_1, P_2) = (0.1, 0.5)$, $(0.3, 0.5)$ and $(0.4, 0.5)$, respectively. Arrival rate is fixed at $\lambda = 0.9$ bits per time slot. We observe from the graphical results that the optimal threshold increases as $P_1$ approaches to $P_2$. This is because for very small $P_1$, the error rate is high, hence $P_2$ is preferred even for smaller queue sizes. However as $P_1$ approaches to $P_2$, the error rates become closer, hence the overflow probabilities come closer. Therefore $P_2$ is preferred only for very high queue sizes.

Second, we consider the effect of changing the arrival rate on the threshold , for a fixed pair of transmission powers $(P_1, P_2) = (0.3, 0.5)$. Figure 4 shows the optimal power control policies corresponding to $\lambda = 0.4$, 0.6 and 0.9. We see that as arrival rate increases, the optimal threshold value decreases. This is natural because as the arrival rate increases, the expected overflow increases, hence high transmission power is preferred in order to decrease the overflow, with the expense of increased energy expenditure.

Third, we consider the effect of changing the discount factor $\alpha$ on the optimal threshold, for a fixed pair of powers $(P_1, P_2) = (0.3, 0.5)$ and arrival rate $\lambda = 0.9$ bits per time slot. Figure 5 shows the optimal power control policies corresponding to $\alpha = 0.90$, 0.94 and 0.97. We see that the optimal threshold decreases as the discount factor increases. Using high power does not bring an advantage to the current overflow, however it is advantageous in decreasing future overflows. Therefore if we increase discount factor, we increase the importance of future time slots, which leads to the preference of high transmission power for smaller queue sizes.

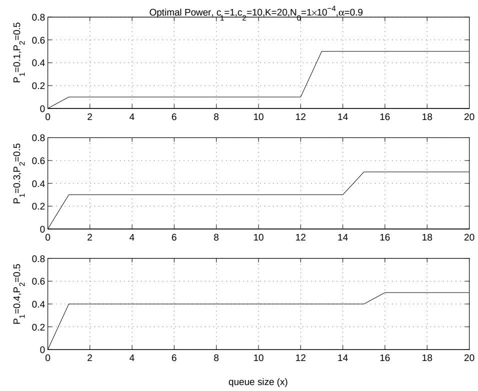


Fig. 3: Optimal power control policy for different power levels.

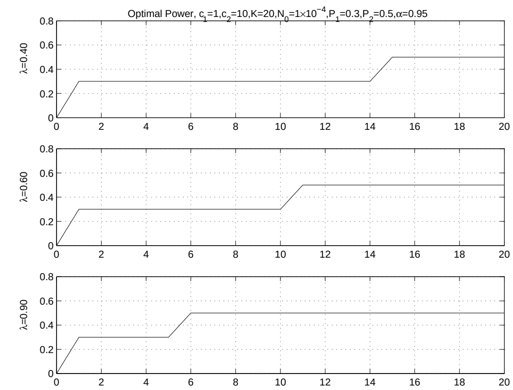


Fig. 4: Optimal power control policy for different arrival rates.

## VI. Conclusions

In this paper we studied on power control over a wireless link with Bernoulli packet arrivals and finite buffer capacity. We considered the problem of jointly minimizing energy expenditure and buffer overflow. We proved that the optimal policy that chooses one of two alternative power levels is of threshold type, that is the optimal policy switches from low power to high at a certain queue size level. Then we exploited this result in order to define a more practical procedure to find the optimal thresholds. Finally we presented some computational results in order to see the characteristics of the threshold policy by changing parameters such as power, arrival rate and discount factor. Simulation results also indicate that the optimal policy is indeed of threshold type. Different schemes of adaptive transmission, such as symbol rate control and modulation level control can also be analyzed within the same framework, which are the directions for future research.

## Appendix:Proof of Theorem 1

We will prove the inequalities by induction.

n=0: We know that for $n = 0$,

$$h_x^0 = g(x,1) - g(x,2) = c_1P_1 + c_2O_x - c_1P_2 - c_2O_x = c_1(P_1 - P_2) < 0$$

$h_x^0$ is constant for all $x \geq 1$. Therefore (14) holds for $x \geq 1$.

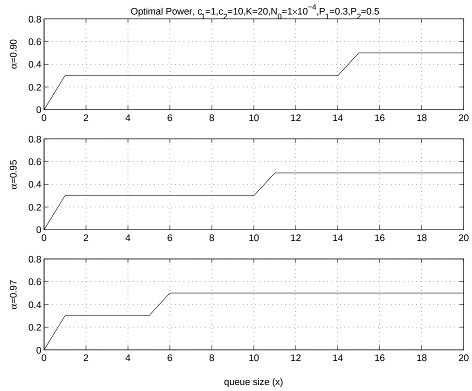


Fig. 5: Optimal power control policy for different discount factors.

Now let's write the expressions for $k_x^0$ and $l_x^0$.

$$k_x^0 = g(x+1,1) - g(x,1) = c_1 P_1 + c_2 O_{x+1} - c_1 P_1 - c_2 O_x = c_2(O_{x+1} - O_x)$$

$$l_x^0 = g(x+1,2) - g(x,2) = c_1 P_2 + c_2 O_{x+1} - c_1 P_2 - c_2 O_x = c_2(O_{x+1} - O_x)$$

Since $O_x$ is a convex function, $k_x^0$ and $l_x^0$ are increasing functions for all $x \geq 1$.

$$V^0(x,i) = c_1 P_i + c_2 O_x \quad and \quad V^0(0,0) = c_2 O_0$$

Since $O_{x+1} - O_x \geq 0$, $k_x^0$ and $l_x^0$ are greater than or equal to zero for $x \geq 1$. Hence all of the inequalities hold for $n = 0$, where $x^0$ is equal to 1.

$1^{st}$ iteration: We first define $a^n(x)$ as the optimal power decision after the $n^{th}$ iteration, when the number in system is $x$. Since $h_x^0 = c_1(P_1 - P_2) < 0$ for all , $a^0(x) = 1$ for all $K \geq x \geq 1$. Therefore from the DP Equation (10):

$$h_1^1 = c_1(P_1 - P_2) + \alpha[\lambda(1-\mu_2)h_2^0 + (1-\lambda)(1-\mu_2)h_1^0 + \lambda(\mu_2 - \mu_1)k_1^0 + (1-\lambda)(\mu_2-\mu_1)(V^0(1,1) - V^0(0,0))] \quad (28)$$

$$h_x^1|_{K>x\geq 2} = c_1(P_1 - P_2) + \alpha[\lambda(1-\mu_2)h_{x+1}^0 + (1-\lambda)(1-\mu_2)h_x^0 + \lambda(\mu_2-\mu_1)k_x^0 + (1-\lambda)(\mu_2-\mu_1)k_{x-1}^0] \quad (29)$$

$$h_x^1|_{x=K} = c_1(P_1 - P_2) + \alpha[(1-\mu_2)h_K^0 + (\mu_2-\mu_1)k_{K-1}^0] \quad (30)$$

For $K \geq x \geq 2$, $h_x^1$ is a weighted sum of functions $h_x^0$, $h_{x+1}^0$, $k_x^0$ and $k_{x-1}^0$, which are all strictly increasing functions for $x \geq 2$. For $x = 1$:

$$h_2^1 - h_1^1 = \alpha(1-\lambda)(\mu_2-\mu_1)[V^0(2,1) - V^0(1,1)) - V^0(1,1) + V^0(0,0)] = -\alpha(1-\lambda)(\mu_2-\mu_1)c_1 P_1 < 0$$

We see that $h_x^1$ is a decreasing function of $x$ for $x = 1$. Let's look at $h_1^1$:

$$\begin{aligned} h_1^1 &= c_1(P_1 - P_2)[1 + \alpha(1-\mu_2)] + \alpha(1-\lambda)(\mu_2-\mu_1)c_1 P_1 \\ &\leq c_1(P_1 - P_2)[2-\mu_2)] + (1-\lambda)(\mu_2-\mu_1)c_1 P_1 \\ &= c_1[(2-\mu_1)P_1 - (2-\mu_2)P_2 - \lambda(\mu_2-\mu_1)P_1] \\ &< 0. \end{aligned}$$

For $x = K$:

$$\begin{aligned} h_K^1 - h_{K-1}^1 &= \alpha c_2(1-\lambda)(\mu_2-\mu_1)k_{K-1}^0 \\ &= \alpha c_2(1-\lambda)(\mu_2-\mu_1)(O_K - O_{K-1}) \\ &= \alpha c_2(1-\lambda)(\mu_2-\mu_1)\lambda > 0 \end{aligned}$$

Therefore $h_x^1$ it is an increasing function for $x \geq 2$. Now we will write the expressions for $k_x^1$:

$$k_1^1 = c_1(O_2 - O_1) + \alpha[\lambda\mu_1 k_1^0 + (1-\lambda)\mu_1(V^0(1,1) - V^0(0,0)) + \lambda(1-\mu_1)k_2^0 + (1-\lambda)(1-\mu_1)k_1^0] \quad (31)$$

$$k_x^1|_{K>x\geq 2} = c_1(O_{x+1} - O_x) + \alpha[\lambda\mu_1 k_x^0 + (1-\lambda)\mu_1 k_{x-1}^0 + \lambda(1-\mu_1)k_{x+1}^0 + (1-\lambda)(1-\mu_1)k_x^0] \quad (32)$$

For $K \geq x \geq 2$, $k_x^1$ is a weighted sum of increasing functions $k_x^1$, $k_{x+1}^1$ and $k_{x-1}^1$, therefore it is an increasing function for $x \geq 2$. For $x < 2$ it is a decreasing function:

$$k_2^1 - k_1^1 = \alpha[(V^0(2,1) - V^0(1,1)) - (V^0(1,1) - V^0(0,0))] = -c_1 P_1 < 0$$

$k_x^1$ is also greater than or equal zero for all $x$, since it is a sum of nonnegative functions for all $x$. Therefore all of the inequalities in the theorem holds for $k_x^1$ in the first iteration. Similar arguments can be used for $l_x^1$, therefore it is omitted. Therefore all inequalities are satisfied for $n = 1$. Let $t_1$ be the threshold for the $1^{st}$ iteration, that is, let $a^1(x)$ be equal to 1 for $0 < x < t_1$ and 2 for $x \geq t_1$.

$n + 1^{st}$ iteration: Assume that after the $n^{th}$ iteration, functions $h_x^n$, $k_x^n$ and $l_x^n$ are increasing for $x \geq x^n$ and the threshold is equal to $t_n$. Let also $a^n(x)$ denote the optimal decision after $n^{th}$ iteration for number in system $x$. $h_x^{n+1}$ can be written as follows:

$$h_x^{n+1} = c_1(P_1 - P_2) + \alpha[\lambda(1-\mu_2)h_{x+1}^n + (1-\lambda)(1-\mu_2)h_x^n + \lambda(\mu_2-\mu_1)V^n(x+1,1) + (1-\lambda)(\mu_2-\mu_1)V^n(x,1) + \lambda\mu_1 W^n(x,1,1) + (1-\lambda)\mu_1 W^n(x-1,1,1) - \lambda\mu_2 W^n(x,2,1) - (1-\lambda)\mu_2 W^n(x-1,2,1)] \quad (33)$$

There are three cases:

1. $x \leq t_n - 1 \longrightarrow a^n(x) = a^n(x-1) = 1 \longrightarrow W^n(x,i,1) = V^n(x,1), i = 1,2,$
   $W^n(x-1,i,1) = V^n(x-1,1), i = 1,2$:
   $$h_x^{n+1} = c_1(P_1 - P_2) + \lambda(1-\mu_2)h_{x+1}^n + (1-\lambda)(1-\mu_2)h_x^n + \lambda(\mu_2-\mu_1)k_x^n + (1-\lambda)(\mu_2-\mu_1)k_{x-1}^n \quad (34)$$
2. $x = t_n \longrightarrow a^n(x) = 2, a^n(x-1) = 1 \longrightarrow$
   $W^n(x,i,1) = V^n(x,2), i = 1,2,$
   $W^n(x-1,i,1) = V^n(x-1,1), i = 1,2$:
   $$h_x^{n+1} = c_1(P_1 - P_2) + \lambda(1-\mu_1)h_{x+1}^n + (1-\lambda)(1-\mu_2)h_x^n + \lambda(\mu_2-\mu_1)l_x^n + (1-\lambda)(\mu_2-\mu_1)k_{x-1}^n \quad (35)$$
3. $x \geq t_n + 1 \longrightarrow a^n(x) = a^n(x-1) = 2 \longrightarrow$
   $W^n(x,i,1) = V^n(x,2), i = 1,2,$
   $W^n(x-1,i,1) = V^n(x-1,2), i = 1,2$:
   $$h_x^{n+1} = c_1(P_1 - P_2) + \lambda(1-\mu_1)h_{x+1}^n + (1-\lambda)(1-\mu_1)h_x^n + \lambda(\mu_2-\mu_1)l_x^n + (1-\lambda)(\mu_2-\mu_1)l_{x-1}^n \quad (36)$$

In all of the three cases, $h_x^{n+1}$ is a sum of functions that are increasing for $x \geq x^n$. But we should also look into the boundaries $x = t_n$ and $t_n + 1$:

$$h_{t_n}^{n+1} - h_{t_n-1}^{n+1} = \lambda(\mu_2 - \mu_1)h_{t_n}^n + \lambda(1-\mu_2)(h_{t_n+1}^n - h_{t_n}^n) + (1-\lambda)(1-\mu_2)(h_{t_n}^n - h_{t_n-1}^n) + \lambda(\mu_2 - \mu_1)(k_{t_n}^n - k_{t_n-1}^n) + (1-\lambda)(\mu_2-\mu_1)(k_{t_n-1}^n - k_{t_n-2}^n) \geq 0 \quad (37)$$

$$h_{t_n+1}^{n+1} - h_{t_n}^{n+1} = \lambda(1-\mu_1)(h_{t_n+2}^n - h_{t_n+1}^n) + (1-\lambda)(1-\mu_2)(h_{t_n+1}^n - h_{t_n}^n) + \lambda(\mu_2-\mu_1)(l_{t_n+1}^n - l_{t_n}^n) + (1-\lambda)(\mu_2-\mu_1)h_{t_n}^n \geq 0 \quad (38)$$

We know that in all three cases, $h_x^n$, $k_x^n$ and $l_x^n$ are all increasing functions for $x \geq x^n$, so we can say that $h_x^{n+1}$ is an increasing function for $x \geq x^n + 1$.

Now we will analyze $k_x^{n+1}$:

$$k_x^{n+1} = c_1(O_{x+1} - O_x) + \lambda\mu_1(W^n(x+1,1,1) - W^n(x,1,1)) + (1-\lambda)\mu_1(W^n(x,1,1) - W^n(x-1,1,1)) + \lambda(1-\mu_1)k_{x+1}^n + (1-\lambda)(1-\mu_1)k_x^n \quad (39)$$

There are four cases:

1. $x \leq t_n - 2 \longrightarrow a^n(x+1) = a^n(x) = a^n(x-1) = 1$
   $W^n(x+1,i,1) = V^n(x+1,1), i = 1,2,$
   $W^n(x,i,1) = V^n(x,1), i = 1,2,$
   $W^n(x-1,i,1) = V^n(x-1,1), i = 1,2:$

$$k_x^{n+1} = c_1(O_{x+1} - O_x) + \lambda\mu_1 k_x^n + (1-\lambda)\mu_1 k_{x-1}^n + \lambda(1-\mu_1)k_{x+1}^n + (1-\lambda)(1-\mu_1)k_x^n \quad (40)$$

2. $x = t_n - 1 \longrightarrow a^n(x+1) = 2, a^n(x) = a^n(x-1) = 1$
   $W^n(x+1,i,1) = V^n(x+1,2), i = 1,2,$
   $W^n(x,i,1) = V^n(x,1), i = 1,2,$
   $W^n(x-1,i,1) = V^n(x-1,1), i = 1,2:$

$$k_x^{n+1} = c_1(O_{x+1} - O_x) + \lambda\mu_1(l_x^n - h_x^n) + (1-\lambda)\mu_1 k_{x-1}^n + \lambda(1-\mu_1)k_{x+1}^n + (1-\lambda)(1-\mu_1)k_x^n \quad (41)$$

3. $x = t_n \longrightarrow a^n(x+1) = a^n(x) = 2,\ a^n(x-1) = 1$
   $W^n(x+1,i,1) = V^n(x+1,2), i = 1,2,$
   $W^n(x,i,1) = V^n(x,2), i = 1,2,$
   $W^n(x-1,i,1) = V^n(x-1,1), i = 1,2:$

$$k_x^{n+1} = c_1(O_{x+1} - O_x) + (1-\lambda)\mu_1(V^n(x,2) - V^n(x-1,1)) + \lambda\mu_1 l_x^n + \lambda(1-\mu_1)k_{x+1}^n + (1-\lambda)(1-\mu_1)k_x^n \quad (42)$$

4. $x = t_n + 1 \longrightarrow a^n(x+1) = a^n(x) = a^n(x-1) = 2$
   $W^n(x+1,i,1) = V^n(x+1,2), i = 1,2,$
   $W^n(x,i,1) = V^n(x,2), i = 1,2,$
   $W^n(x-1,i,1) = V^n(x-1,2), i = 1,2:$

$$k_x^{n+1} = c_1(O_{x+1} - O_x) + \lambda\mu_1 l_x^n + (1-\lambda)\mu_1 l_{x-1}^n + \lambda(1-\mu_1)k_{x+1}^n + (1-\lambda)(1-\mu_1)k_x^n \quad (43)$$

We again know that in all four cases, $h_x^n$, $k_x^n$ and $l_x^n$ are all increasing functions for $x \geq x^n$, so we can say that $k_x^{n+1}$ is an increasing function for $x \geq x^n + 1$. The same arguments can be used for $l_x^{n+1}$ and can be seen that it is also an increasing function for $x \geq x^n + 1$, which completes the proof.

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